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Do Solar System Tests Permit Higher Dimensional General Relativity?

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Abstract We perform a survey whether higher dimensional Schwarzschild space-time is compatible with some of the solar system phenomena. As a test we examine four well known solar system effects, viz., (1) Perihelion shift, (2) Bending of light, (3) Gravitational redshift, and (4) Gravitational time delay. It is shown, under a N-dimensional solutions of Schwarzschild type very narrow class of metrics, that the results related to all these physical phenomena are mostly incompatible with the higher dimensional version of general relativity. We compare all these restricted results with the available data in the literature.

Keywords Gravitation · Solar system · Celestial mechanics

1 Introduction

Einstein's general relativity (GR) received first remarkable success due to the observational confirmation of two solar system effects, firstly, contribution to perihelion shift of 43 arcsec per century as curvature effect of space-time, and secondly, total solar eclipse in the year 1919 which admits the relativistic value 1.75 arcsec as obtained by Einstein which is also due to the effect of curvature. Probably these observational boldness and the sublime structure of GR inspired Born [1] to state that:

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The theory appeared to me then, and it still does, the greatest feat of human thinking about nature, the most amazing combination of philosophical penetration, physical intuition, and mathematical skill. It appealed to me like a great work of art....

However, history tells us that Einstein's theory did not achieve any instant success; he presented it in November 1915, but did not win unqualified acceptance until after the eclipse expedition results were announced in November 1919. This triumphs of GR are obviously based on the principles of equivalence and general covariance. As GR in 4-dimensions gives a perfectly satisfactory theory of gravitation so it was the desire to unify gravity with other interactions that prompted people to start thinking about higher dimensions (as early as 1914, but especially from the 1920s on). On the other hand, the extension of GR by the inclusion of dimensions beyond four were initiated by investigators in connection to the studies of early Universe (though that connection was not made until the 1980s) [2]. It is commonly believed that the four-dimensional present space-time is the compactified form of manifold with higher dimensions (HD). This self-compactification of multidimensions have been thought of by several researchers [3, 4] in the area of grand unification theory as well as in superstring theory. It is worth while to note here that Barrow [5] has examined the role played by the dimensions of space and space-time in determining the form of various physical laws and constants of Nature. To do so he has exploited the concept of fractal dimension under Kaluza-Klein theories obtained by dimensional reduction from higher dimensional gravity or supergravity theories.

In the Kaluza-Klein gravitational theory with higher dimensions, therefore, it is a common practice to show that extra dimensions are reducible to lower one, specially in four-dimension which was associated with some physical processes. Interestingly, mass have been considered as the fifth dimension [6–10] in the case of five-dimensional Kaluza-Klein theory. Fukui [7] suggested that expansion of the Universe follows by the percolation of radiation into 4-dimensional space-time from the fifth dimensional mass. Ponce de Leon [10] argued that the rest mass of a particle, perceived by an observer in four-dimension, varies as a result of the five-dimensional motion along the extra direction and in the presence of elctromagnetic field is totally of elctromagnetic origin which has confirmed by Ray [11]. On the other hand, it have been shown by many investigators [12, 13] that within the Kaluza-Klein inflationary scenario of HD a contraction of the internal space causes the inflation of the usual space. There are cases in FRW cosmologies where the extra dimensions contract as a result of cosmological evolution [14]. In the solution to the vacuum field equations of GR in $4+1$ dimensions Chodos and Detweiler [15] have shown that it leads to a cosmology which at the present epoch has $3+1$ observable dimensions in which the Einstein-Maxwell equations are obeyed.

There are observationally $N > 4D$ solutions available in the literature [16–19] though with different conclusions. In fact, we would like to emphasize that here we are dealing with a very narrow class of metrics and that there are several other classes of metrics for $N > 4D$ which agree with all of the classical tests. One example is that of the canonical metric of Mashhoon et al. [18] which has also been used by Tavakol et al. [20]. Other relevant papers in this line are by Seahra et al. [21] and Wesson et al. [19] and also that by Ponce de Leon [17] on non-Schwarzschild star exterior solutions.

Under these theoretical background, therefore, now-a-days people have started to think of the higher dimensional influence on GR, more precisely, whether within the framework of higher dimensional GR the same type of solar system tests would yield the same results. Actually, it has two-fold intentions: firstly, if the results are positive then the higher dimensional version of GR will prove itself as an extended viable theory of gravitation, and secondly, if negative then there is no need of higher dimensional GR at all. Motivated by this, therefore, in a recent work[1] Liu and Overduin [25] argued that to test the theory involving the motion of test particles in the field of a static spherically-symmetric mass like the Sun or the Earth would be most straightforward. Kagramanova et al. [26] have investigated Solar system effects in Schwarzschild-de Sitter space-time and estimated the values for the cosmological parameter Λ. In a similar line of thinking Iorio [27, 28] attempted to investigate secular increase of the Astronomical Unit, perihelion precessions and planetary motions as tests of the Dvali-Gabadadze-Porrati multidimensional braneworld scenario.

In this connection, however, it is to be noted here that among the recent investigations by Liu and Overduin [25], along with those of Lim et al. [29] and Kalligas et al. [30], are limited to five-dimensional soliton-like space-time only. Therefore, our present attempt is to study more general cases under a spherically symmetric Schwarzschild-like space-time with N number of dimensions where $N = D + 2$ such that $D \geq 2$. In this context we discuss the following four cases involved in the solar system experiments to examine the viability of GR with HD, viz., (1) Perihelion shift, (2) Bending of light, (3) Gravitational Red-shift, and (4) Gravitational time delay. In this regard we would like to mention here that there is a general statement (see e.g. the paper by Barrow [5], or Hackmann et al. [31]) that there are no bound orbits in Tangherlini space-time [24] or actually to be more precise: There are no quasi-periodic bound orbits. There are bound orbits ending in the singularity and there are unbound orbits arriving from and leaving to infinity. However, even then we are trying to solve the geodesic equation for bound orbits to conclude that higher dimensions are not compatible with observations. Actually in our present investigations we would like to reasonably show, under a higher dimensional solutions of Schwarzschild type restricted class of metrics, that most of these solar system phenomena do not allow dimensions beyond 4.

2 Mathematical Formulation

We consider the space-time which has the topology of $R^2 \times S^D$. The action for this model is $I = \int (-g)^{\frac{1}{2}} [R_{D+2} + L_m] d^{D+2}X$ where, R_{D+2} is the curvature scalar in $D+2$ dimensional space-time and L_m stands Lagrangian for the matter distribution (for vacuum space-time L_m identically zero). Note that here the topology being $R^2 \times S^D$ we use the notation $N = D+2$. In this work we have performed the general calculations with $N = D + 2$ dimensions and after that we have discussed several subcases assuming $D = 2, 3, 4$ etc. It is also to be noted that for the subcase $D = 2$ (i.e., standard GR with no extra dimensions), our results coincide with the results existing in literature.

Let us then consider a spherically symmetric metric which represents a generalized Schwarzschild space-time with higher dimensions [32]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_D{}^2, \tag{1}$$

where r is a radial coordinate and f is a function of r only.

The above equation (1) was introduced and studied in detail by Myers and Perry in 1986. This metric, in which all the compact extra dimensions are spherically symmetric, is one of many possible generalizations of the $4D$ Schwarzschild metric to higher dimensions.

[1] We would like to add here that though the subject of solar-system tests of higher-dimensional extensions of GR goes back as far as Heckmann, Jordan and Fricke [22], Kühnel and Schmutzer [23] and Tangherlini [24] but the motivation of the present investigation is based on the recent works only.

But it is not obvious that compact extra dimensions must have spherical symmetry, since these dimensions are not accessible to observation. Dobiasch and Maison [33] considered an alternative with a flat internal space. Another kind of generalization allows for cross-terms between compact and macroscopic dimensions, as considered by Gibbons and Wiltshire [34] and that by Yoshimura [35] which allows part of the metric tensor to depend explicitly on the extra coordinates.

Now, the line element $d\Omega_D{}^2$ on the unit D-sphere is given by

$$d\Omega_D{}^2 = d\theta_1{}^2 + \sin^2\theta_1 d\theta_2{}^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3{}^2 + \cdots + \prod_{n=1}^{D-1}\sin^2\theta_n d\theta_D{}^2$$

$$\equiv dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + \cdots + dx_D{}^2 \tag{2}$$

where x_1, x_2, $x_3, \ldots, x_D$ are independent coordinates on the surface of the unit D-sphere with $\Omega_D = 2[\pi^{(D+1)/2}]/[\Gamma(D+1)/2]$. Also, according to Einstein equations we can write $f(r) = 1 - \mu/r^{D-1}$ [24] with the constant of integration $\mu = 16\pi GM/Dc^2\Omega_D$. It is to be noted here that the radius vector r is considered positive if measured from the pole along the line bounding the vectorial angle and negative if measured in the opposite direction. Suppose PO is produced to P' making OP' equal to OP, then its polar coordinates are $(-r, \theta)$. This can also represented as $(-r, \pi + \theta)$. Thus we see that the same radius vector may be positive or negative according to the vectorial angle to which it is associated.

Now, in principle, in Lagrangian mechanics the trajectory of an object is derived by finding the path which minimizes the action, a quantity which is the integral of the Lagrangian over time. So, in connection to the solar system problem we would like to adopt the higher dimensional Lagrangian which can be written as

$$L = T - V = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta_1}^2 + r^2\sin^2\theta_1\dot{\theta_2}^2 + \cdots + r^2\prod_{n=1}^{D-1}\sin^2\theta_n\dot{\theta_D}^2. \tag{3}$$

Here dot over any parameter implies differentiation with respect to the affine parameter τ.

It is known that the gravitational field is isotropic and hence there is conservation of angular momentum. So geodesic of the particles (either massive planets or massless photons) are planar. Without loss of generality, we can choose our coordinates in such a way that this plane is the equatorial plane by keeping fixed $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{D-1} = \pi/2$. Therefore, if we take a cross-section with this fixed angular prescription so that $\dot{\theta_i} = 0, i = 1, 2, 3, \ldots, D-1$ then the Lagrangian takes the form

$$L = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta_D}^2 \tag{4}$$

with light-like particle photon, $L = 0$ and for any time-like particle, $L = 1$. In this connection we would like to mention that since we are dealing with the space-time comprising the topology $R^2 \times S^D$, so similar to space-time with topology $R^2 \times S^2$, we can set $L = 0$ for photon and $L = 1$ for massive particle.

Therefore, in terms of the generalized coordinates q_i and generalized velocities $\dot{q}_i$, the standard Euler-Lagrange equations are

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{5}$$

By assuming $f\dot{t} = E$ and $r^2\dot{\theta_D} = \pounds$, where E and $\pounds$ are the energy and angular momentum of the particle respectively, such that $\dot{t} = E/f$ and $\dot{\theta_D} = \pounds/r^2$ and hence with these notations equation (4) becomes

$$L = -\frac{E^2}{f} + \frac{\dot{r}^2}{f} + \frac{\pounds^2}{r^2} \tag{6}$$

which, after simplification, can be written in the following forms

$$\dot{r}^2 = Lf + E^2 - \frac{\pounds^2 f}{r^2} \tag{7}$$

and

$$\frac{1}{r^4}\left(\frac{dr}{d\theta_D}\right)^2 = \frac{Lf}{\pounds^2} + \frac{E^2}{\pounds^2} - \frac{f}{r^2}. \tag{8}$$

Again, by substituting $\theta_D = \phi$ and $r = 1/U$ in (8), one can write

$$\left(\frac{dU}{d\phi}\right)^2 = \frac{Lf}{\pounds^2} + \frac{E^2}{\pounds^2} - fU^2. \tag{9}$$

Now, if we write (7) in the form

$$\left(\frac{dr}{dt}\right)^2 = \frac{Lf^3}{E^2} + f^2 - \frac{\pounds^2 f^3}{E^2 r^2} \tag{10}$$

then one can easily observe that $\frac{dr}{dt}$ vanishes at $r = r_0$ of the closest approach to the sun. This at once yields the relationship between momentum and energy of the particle as follows: $\pounds^2/E^2 = r_0^2/f_0$, where $f_0 = f(r = r_0)$. Hence, the equation of photon becomes

$$\left(\frac{dr}{dt}\right)^2 = f^2 - \frac{f^3 r_0^2}{f_0 r^2}. \tag{11}$$

Thus, the time required for light to travel from r_0 to r can be expressed as

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr}{\left[f^2 - \frac{f^3 {r_0}^2}{f_0 r^2}\right]^{1/2}} \tag{12}$$

which is given here for radial motion only.

3 Solar System Tests for Higher Dimensional GR

3.1 Perihelion Shift

Following (8), motion of planet in the sun's gravitational field can be written as

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{f}{\pounds^2} + \frac{E^2}{\pounds^2} - \frac{f}{r^2}. \tag{13}$$

For $r = 1/U$, we have

$$\frac{d^2U}{d\phi^2} + U = \mu(D+1)U^D + \frac{\mu}{\pounds^2}(D-1)U^{D-2}. \tag{14}$$

The solution to (14) is then given by the following cases:

(i): $D = 2$

By the use of successive approximation (taking $\mu = 0$ as zeroth approximation) we get the solution to the above equation (14) in the form

$$U = \frac{1}{l}(1 + e\cos\phi) \tag{15}$$

where $l = \pounds^2/\frac{GM}{c^2}$. Obviously, the trajectory of test particle, i.e., planet is elliptical.

Substituting this on the right hand side for U, we get

$$U = \frac{1}{l}[1 + e\cos(\phi - \omega)] \tag{16}$$

with $\omega = (3GM/c^2l)\phi$. Therefore, time period for the planet is $T = 2\pi(3GM/c^2l)$ and the average precession can be obtained as $n = 6\pi GM/c^2lT = 43.03$ arcsec per century (where $l = 5.53 \times 10^{12}$ cm, $GM/c^2 = 1.475 \times 10^5$ cm and one century $= 415T$). This value of Mercury's perihelion precession rate is very close to some of the available data which are 43.11 ± 0.21 and 42.98 arcsec per century respectively as obtained by Shapiro et al. [36] and Liu and Overduin [25]. However, our results are different for all five dimensional cases with Liu and Overduin [25] as because we have assumed the space-time with topology $R^2 \times S^3$ whereas they have assumed the space-time with topology $R^2 \times S^2 \times R^1$.

(ii): $D = 3$

The equation of motion in this case can be written as

$$\frac{d^2U}{d\phi^2} + U\left(1 - \frac{2\mu}{\pounds^2}\right) = 4\mu U^3. \tag{17}$$

The solution is given by

$$U = U_0\cos(\beta\psi) + U_1\cos(3\beta\psi) \tag{18}$$

where $\psi = a\phi$, $U_1 = -(2\mu/16a)U_0^2 \ll U_0$ and $\beta^2 = 1 - (3\mu/a)U_0^2$ with $a = 1 - 2\mu/\pounds^2$. Hence, one can observe that the path is not elliptical in shape.

(iii): $D = 4$

The solution of (14) related to motion of planet in this case is

$$\begin{aligned} U = {} & \frac{\cos\phi}{r_0} + \frac{5\mu}{r_0^4}\left[\frac{3}{8} - \frac{1}{6}(2\cos^2\phi - 1)\right. \\ & \left. - \frac{1}{120}(8\cos^4\phi - 8\cos^2\phi - 1)\right] \\ & + \frac{3\mu}{\pounds^2 r_0^2}\left[\frac{1}{2} - \frac{1}{3}(2\cos^2\phi - 1)\right]. \end{aligned} \tag{19}$$

Again, the path is no longer elliptical but periodic bound. This suggests that orbit of the planet is not elliptical in the sun's gravitational field (space-time with topology $R^2 \times S^D$) for $D \geq 3$. Actually, the unique form of the equation which represents the ellipse in polar coordinate is $\frac{l}{r} = 1 + e\cos\theta$. So, other forms ((18) and (19)) should not represent the ellipses. It is to be emphasized here that if the orbits are ellipses, then there is no perihelion shift. One actually needs non-elliptical orbits to obtain a perihelion shift. What one needs is a bound orbit and that is what one does not obtain in higher dimensions.

3.2 Bending of Light

Now we would like to observe how higher dimensional version of general relativity do respond on the effect of light bending. Let us, therefore, start with (9) which now reads

$$\left(\frac{dU}{d\phi}\right)^2 = \frac{E^2}{\pounds^2} - U^2(1 - \mu U^{D-1}). \tag{20}$$

The above equation can be written in the suitable form as

$$\frac{d^2U}{d\phi^2} + U = \frac{\mu(D+1)}{2} U^D. \tag{21}$$

Now, we solve the equation by successive approximation, starting with the straight line (path without gravitating body) as zeroth approximation such that $U = \cos\phi/R_0$ where $\phi = 0$ is the point P of nearest approach to the Sun's surface. Ideally, R_0 would be the solar radius.

Substituting this on the right hand side of (21) for U, we get

$$\frac{d^2U}{d\phi^2} + U = \frac{\mu(D+1)}{2R_0^D} \cos^D\phi. \tag{22}$$

The solution of the above equation (22) is then given by for the following cases:

Case I: $D = \text{even} = 2n$

Let us consider the case when D is even and takes the value $2n$. For this particular situation the exact solution to (22) can be given as

$$U = \frac{\cos\phi}{R_0} + \frac{\mu(2n+1)}{2^{2n}R_0^{2n}}\left[\frac{\cos 2n\phi}{-4n^2+1} + \frac{{}^{2n}C_1\cos(2n-2)\phi}{-(2n-2)^2+1} + \cdots + \frac{{}^{2n}C_{n-1}\cos 2\phi}{-2^2+1}\right] + \frac{\mu(2n+1)^{2n}C_n}{2^{2n+1}R_0^{2n}}. \tag{23}$$

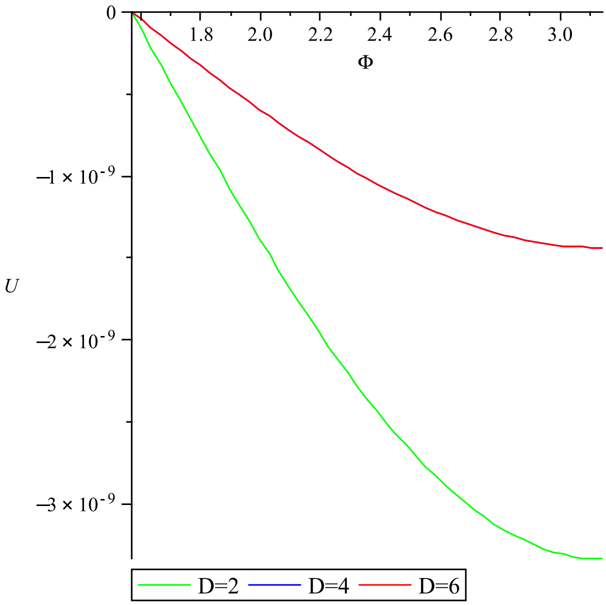
(i): $n = 1$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{GM}{R_0^2c^2}(2 - \cos^2\phi). \tag{24}$$

For $U = 0$, we get $\cos\phi = -0.4244302380 \times 10^{-5}$ where the values for the constants are taken as follows: $c = 2.997925 \times 10^8$ m/sec, $G = 6.67323 \times 10^{-11}$ SI Unit, $M = 1.9892 \times 10^{30}$ Kg and $R_0 = 6.95987 \times 10^8$ m.

Fig. 1 The plot U vs. ϕ for $D = 2, 4, 6$



Here the net deflection of the ray is given by

$$\Delta\phi = 1.741300716 \text{ arcsec}. \tag{25}$$

This result is in agreement with the experimental result of observed deflection of light by the Sun.

(ii): $n = 2$

Here

$$U = \frac{\cos\phi}{R_0} - \frac{\mu}{6R_0^4}\cos^4\phi - \frac{2\mu}{3R_0^4}\cos^2\phi + \frac{4\mu}{3R_0^4}. \tag{26}$$

For $U = 0$, we get $\cos\phi = 0.44741136 \times 10^{-2}$ so that $\Delta\phi = -0.8945 \times 10^{-2}$ arcsec.

(iii): $n = 3$

For the value $n = 3$, we get

$$U = \frac{\cos\phi}{R_0} - \frac{7\mu}{64R_0^6}\left[\frac{1}{35}(32\cos^6\phi - 48\cos^4\phi + 8\cos^2\phi - 1) + \frac{2}{5}(8\cos^4\phi - 8\cos^2\phi + 1) + 5(2\cos^2\phi - 1)\right] + \frac{35\mu}{32R_0^6} \tag{27}$$

Here $U = 0$ yields $\cos\phi = -0.2255226204 \times 10^{-41}$ so that $\Delta\phi = -0.1264 \times 10^{-2}$ arcsec. This value of $\Delta\phi$ expresses the angle of surplus rather than angle of deficit [37, 38]. We plot U vs. ϕ for all $D = 2, 4, 6$ in the Fig. 1.

Case II: $D = \text{odd} = 2n-1$

$$U = \frac{\cos\phi}{R_0} + \frac{\mu n}{2^{2n-2}R_0^{2n-1}}\left[\frac{\cos(2n-1)\phi}{-(2n-1)^2+1} + \frac{{}^{2n-1}C_1\cos(2n-3)\phi}{-(2n-3)^2+1} + \cdots\right] + \mu n\frac{{}^{2n-1}C_{n-1}}{2^{2n-2}R_0^{2n-1}}\frac{\phi\sin\phi}{2} \tag{28}$$

(i): $n = 2$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{2\mu}{R_0^3}\left[\frac{3\phi}{8}\sin\phi - \frac{1}{32}\cos 3\phi\right]. \tag{29}$$

For $U = 0$, we get $\cos\phi = -5.176406943 \times 10^{-15}$. Here the net deflection of the ray is given by

$$\Delta\phi = -0.1264 \times 10^{-2}\ \text{arcsec}, \tag{30}$$

which is nothing but angle of surplus.

(ii): $n = 3$

In this subcase

$$U = \frac{\cos\phi}{R_0} + \frac{3\mu}{R_0^5}\left[\frac{5\phi}{16}\sin\phi - \frac{5}{128}\cos 3\phi - \frac{1}{384}\cos 5\phi\right]. \tag{31}$$

For $U = 0$, we get $\cos\phi = -6.122904952 \times 10^{-33}$. Here the net deflection of the ray is given by

$$\Delta\phi = -0.1264 \times 10^{-2}\ \text{arcsec}, \tag{32}$$

which is again angle of surplus. We plot U vs. ϕ for all $D = 3, 5$ in the Fig. 2.

Thus, our observation is that all cases for $D > 2$ i.e. above 4-dimension, we get angles of surplus. We also observe that when $D = 2$, viz., the total dimensions, $N = D + 2 = 4$ then only the result does agree with the observational data and $D > 2$, i.e., $N > 4$ is not compatible with solar system (see Table 1). It can be noted, from the Figs. 1 and 2, that all the trajectories of the light rays almost same for all D due to the factor $\cos\phi/R_0$ which is the dominating one. The trajectory of light will show different graph for large D. In this regard it is also to be mentioned here that we have rechecked our calculations in the bending of light section. We see that the results (i.e. net reflection of light) up to 10 digits are same for $D \geq 3$. If we increase the dimension more than five the effect on the deflection of light (i.e. net reflection of light for $D = 3$) will be negligible.

In connection to our results it is interesting to note that an analysis of large amount of Very Long Baseline Interferometry (VLBI) observations has shown that the ratio of the actual observed deflections to the deflections predicted by general relativity is very close to unity (e.g., 0.9996 ± 0.0017 [39], 0.99994 ± 0.00031 [40], 0.99992 ± 0.00023 [41]).

3.3 Gravitational Redshift

GR predicted that the frequency of the light would be affected due to gravitational field and is observable as a shift of spectral lines towards the red end of the spectrum. Pound-

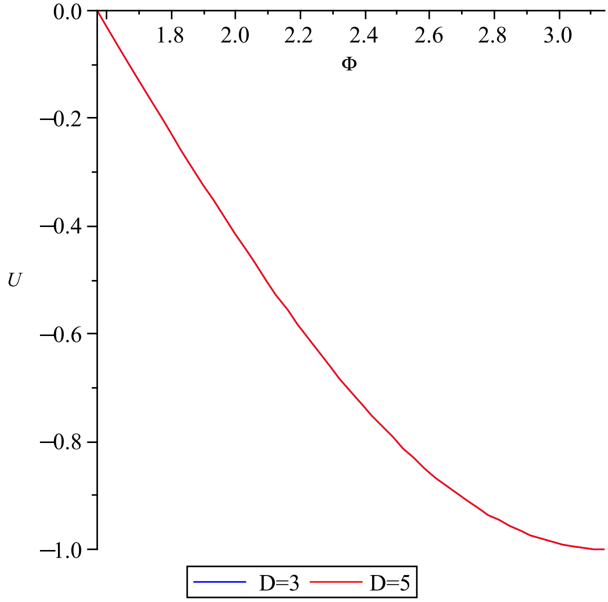


Fig. 2 The plot U vs. ϕ for $D = 3, 5$

Table 1 Deflection of starlight for HD-models

Dimensions (N)	Deflection ($\Delta\phi$) (arcsec)
4	1.741300716
5	-0.1264×10^{-2}
6	-0.8945×10^{-2}
7	-0.1264×10^{-2}
8	-0.1264×10^{-2}

Rebka-Snider [42–44] confirmed this effect through their precision test, sometimes known as Harvard Tower Experiment. In their first test they measured the redshift experienced by a 14.4 Kev γ-rays from the decay of Fe^{57} for a height of 22.5 meter tower and found $z = 2.57 \pm 0.26 \times 10^{-15}$.

Now, as usual, gravitational redshift for the solar system can be defined as

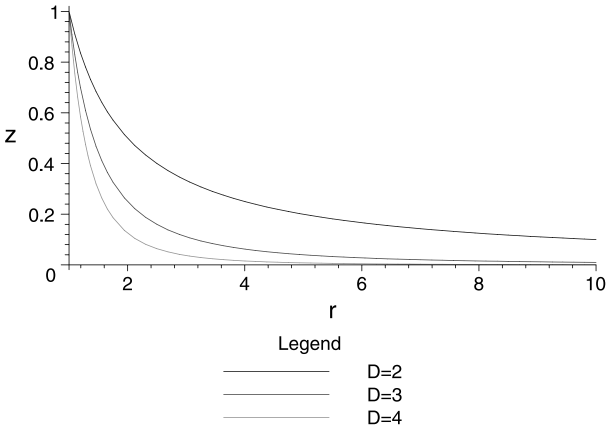
$$z = \frac{\Delta\gamma}{\gamma} = \left[\frac{g_{tt}(R^\star)}{g_{tt}(R)}\right]^{1/2} - 1 \tag{33}$$

where R is the radius of the sun and $R^\star$ is the radius of the earth's orbit around sun.

Therefore, in view of the given HD line element (1), the metric tensors involved in the above equation (33) reduce to

$$\left[\frac{g_{tt}(R^\star)}{g_{tt}(R)}\right]^{1/2} = \left[\frac{1 - \frac{\mu}{R^{\star D-1}}}{1 - \frac{\mu}{R^{D-1}}}\right]^{1/2}$$
$$\cong \left[1 + \frac{\mu}{2R^{D-1}} - \frac{\mu}{2R^{\star D-1}}\right]. \tag{34}$$

Fig. 3 The plot redshift z vs. radial distance r for different dimensions



By substituting the expressions of (34) in (33) for the assumption $R^\star \gg \mu$, we get

$$z = \frac{\Delta\gamma}{\gamma} = \frac{\mu}{2R^{D-1}} \tag{35}$$

where the integration constant μ is related to the mass function. Thus, in the Sun-Earth system we observe that for the usual 4-dimensional case ($D = 2$), gravitational redshift becomes $z \sim 2.12 \times 10^{-6} = z_2$ (say). Therefore, for $D > 2$, $z < z_2$ which indicates that as dimension increases the redshift gradually decreases (see Fig. 3). It can also be observed that redshift gradually increases with mass (since $z \propto \mu = 16\pi GM/Dc^2\Omega_D$, when radial distance and dimensions remain fixed in (35)). However, if μ is kept fixed then the redshift decreases with D. Thus, it seems that dimension acts as inversely proportional to mass of the gravitating body.

3.4 Gravitational Time Delay

Gravitational time delay, also known as Shapiro time delay which was reported by Shapiro [45] is basically the effect of radar signals passing near a massive object take slightly longer time for a round trip as measured by the observer than it would be in the absence of the object there. To proceed on towards the 'Fourth Test of General Relativity' let us consider (12) in the form

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr}{(1-\frac{\mu}{r^{D-1}})\big[1-\frac{1-\frac{\mu}{r^{D-1}}}{1-\frac{\mu}{r_0^{D-1}}}(\frac{r_0}{r})^2\big]^{1/2}} \tag{36}$$

which, after simplification, yields

$$t(r, r_0) = \int_{r_0}^{r} \left(1-\frac{r_0^2}{r^2}\right)^{-1/2} \times \left[1+\frac{\mu}{r^{D-1}}+\frac{\frac{\mu}{2}(r_0^{D-1}-r^{D-1})}{(r_0^2-r^2)r^{D-1}r_0^{D-3}}\right]dr. \tag{37}$$

Hence, transit time of the light ray from Mercury to Earth can be given by

$$t = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[1 + \frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2) r^{D-1} r_0^{D-3}}\right] dr + \int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[1 + \frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2) r^{D-1} r_0^{D-3}}\right] dr. \tag{38}$$

In the absence of the gravitational field (viz., $\mu = 0$) one can get

$$t_0 = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} dr + \int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} dr. \tag{39}$$

Hence, time delay for a round trip is

$$\begin{aligned} \Delta t &= 2(t - t_0) \\ &= 2\int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2) r^{D-1} r_0^{D-3}}\right] dr \\ &\quad + 2\int_{r_0}^{r_2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2) r^{D-1} r_0^{D-3}}\right] dr. \end{aligned} \tag{40}$$

Let us consider

$$I = \int_{r_0}^{r_1} \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2} \left[\frac{\mu}{r^{D-1}} + \frac{\frac{\mu}{2}(r_0^{D-1} - r^{D-1})}{(r_0^2 - r^2) r^{D-1} r_0^{D-3}}\right] dr. \tag{41}$$

The solution to (41) is then given by for the following cases:

(i): $D = 2$

In this case the integral in (41) becomes

$$I = \mu \ln\left[r + \sqrt{r^2 - r_0^2}\right]_{r_0}^{r} + \frac{\mu}{2}\left[\sqrt{\frac{r - r_0}{r + r_0}}\right]_{r_0}^{r} \tag{42}$$

so that the time delay for a round trip can be given as

$$\Delta t = \frac{4GM}{c^2} \ln\left[\frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2}\right] + \frac{2GM}{c^2}\left[\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_2 - r_0}{r_2 + r_0}}\right]. \tag{43}$$

If, however, $r_0 \ll r_1$ and $r_0 \ll r_2$, then

$$\Delta t = \frac{4GM}{c^2}\left[1 + \ln \frac{4 r_1 r_2}{r_0^2}\right]. \tag{44}$$

The above expression for radar echo delay is in accordance with the standard literature [46] when the Schwarzschild space-time is of usual four-dimensional entity and provides an amount 240 μsec as the maximum excess time delay for the Earth-Mercury system.

(ii): $D = 3$

Here

$$I = \frac{3\mu}{2r_0} \sec^{-1}\left(\frac{r}{r_0}\right). \tag{45}$$

Hence, the time delay in this case becomes

$$\Delta t = \frac{3\mu}{r_0}\left[\sec^{-1}\left(\frac{r_1}{r_0}\right) + \sec^{-1}\left(\frac{r_2}{r_0}\right)\right]. \tag{46}$$

Let us consider that either $x = (r_1/r_0) \gg 1$ or $x = (r_2/r_0) \gg 1$ so that, after neglecting the higher order terms like $1/x^3$, $1/x^5, \ldots$ etc. we get

$$\sec^{-1} x \cong \frac{\pi}{2} - \frac{1}{x} \tag{47}$$

so that

$$\Delta t \cong \frac{4GM}{c^2}\left[\frac{1}{r_0} - \frac{1}{\pi}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right]. \tag{48}$$

(iii): $D = 4$

For this case we have

$$I = \frac{3GM}{2\pi c^2}\left[\frac{\sqrt{r^2 - r_0^2}}{r {r_0}^2} + \frac{1}{r_0^2}\sqrt{\frac{r - r_0}{r + r_0}}\right]_{r_0}^{r}. \tag{49}$$

Therefore, the expression for time delay becomes

$$\Delta t = \frac{3GM}{\pi c^2 r_0^2}\left[\frac{\sqrt{r_1^2 - r_0^2}}{r_1} + \frac{\sqrt{r_2^2 - r_0^2}}{r_2} + \sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_2 - r_0}{r_2 + r_0}}\right]. \tag{50}$$

Thus, from the above case studies one can observe that the maximum time delay will occur when $D = 2$, i.e., for the usual 4-dimensional Schwarzschild space-time. Time delay decreases due to increase of dimensions. This can be shown easily by assuming $r_0 \ll r_1$ and $r_0 \ll r_2$.

4 Conclusions

Our analytically performed solar system tests for GR with HD can be summarized as follows:

1. Perihelion shift: In $4D$ our result exactly coincides with that of Einstein's predicted value with an elliptical path followed by the planet Mercury. As we go increase on dimensions the paths just become unbound orbits and hence HD do not work at all.

2. Bending light: Here also our theoretical result is in good agreement with the experimental result 1.741300716 arcsec which become enormously different with an angle of surplus value −0.00948825313 arcsec $D > 2$.
3. Gravitational redshift: We observe that in the 4-dimensional case gravitational redshift becomes $z \sim 2.12 \times 10^{-6}$ in the Sun-Earth system. However, for $D > 2$ redshift gradually decreases with the increase of dimensions such that dimension acts as inversely proportional to mass of the gravitating body. It can also be observed that for constant radial distant and dimension the redshift gradually increases with the mass of the planets.
4. Gravitational time delay: It is seen from the present investigation that radar echo delay is as usual in the case of $4D$ and decreases with increase of dimensions.

In this connection it is to be noted here that the most recent laboratory experiments show no deviation from the Newton's law up to the scale of 50 µm [47]. As on 'weak field approximation' one can get the Newtonian limit of the general relativistic theory, so this result was to some extent expected from our investigations also or in other word this experimental result confirms our theoretically performed Solar system tests. Therefore, in a nutshell, our overall observation regarding HD realm of GR is, in general, similar to that of Liu and Overduin [25] which is as follows: "… the existence of small but potentially measurable departures from the standard $4D$ Einstein predictions".

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